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LETTER TO THE EDITOR

Enumeration study of self-avoiding random surfaces

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Abstract. We employ exact enumeration methods to study a number of configurational properties of self-avoiding random surfaces embedded in a three-dimensional simple cubic lattice. Self-avoiding surfaces are defined as a connected set of plaquettes in which no more than two plaquettes may meet along a common edge, and in which no plaquette can be occupied more than once. Based on enumerating surfaces containing up to 10 plaquettes, we find: (a) the number of n-plaquette surfaces, c_n, varies as $\mu^n n^{\gamma-1}$, with $\mu = 13.2 \pm 0.2$ and $\gamma = 0.22 \pm 0.06$, (b) the average number of perimeter edges of n-plaquette surfaces, $\langle p_n \rangle$, varies linearly with n, and (c) the mean-square radius of gyration of n-plaquette surfaces, $\langle R_g^2(n) \rangle$, varies as $n^{2\nu}$, with $2\nu = 1.075 \pm 0.05$.

In this letter, we consider a number of simple configurational properties of self-avoiding random surfaces (SASs). A SAS is defined as a connected set of elementary plaquettes on a regular lattice in which at most two plaquettes can meet along a common edge. In addition, the surface can occupy each plaquette only once, thereby imposing an excluded-volume constraint. Such a model represents a natural generalisation of the self-avoiding random walk model to a situation where planar elementary units are used to build an object which is topologically two dimensional.

At present, there is a limited amount of information on the properties of SASs. Most of the available results are confined to the properties of *closed* SASs, defined as a self-avoiding surface with no perimeter, because they arise naturally in the high-temperature expansion of lattice gauge theories (see, e.g., Drouffe *et al* (1979) and references therein; for numerical simulation results of closed surfaces, see, e.g., Sterling and Greensite (1983) and Billoire *et al* (1984)). In addition, there have been studies of *open* SASs in which free perimeters may exist (see, e.g., Parisi 1979, Durhuus *et al* 1983, Maritan and Stella 1984), and also of *random* surfaces in which the excluded-volume constraint is removed (Duplantier 1984, Gross 1984). In addition to the purely intrinsic interest of the geometry of surfaces, the SAS model may be relevant to a number of physical problems governed by interfacial phenomena, such as wetting and melting.

Because of the relatively large number of degrees of freedom for surfaces and the corresponding analytical and numerical difficulties, several basic issues are still not resolved. For example, there still exists some controversy about the connection between free random surfaces and the mean-field limit of SASs. Arguments have been presented for both a finite upper critical dimension, d_c, equal to eight for SASs and a fractal dimension of 4 for SASs above the upper critical dimension (Parisi 1979), and also for

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$d_c = \infty$ and an infinite fractal dimensionality of free random surfaces (Duplantier 1984, Gross 1984). The rather incomplete understanding of SASs is one of the motivations for the present study. A second motivating factor is the similarity between the questions that can be asked for the self-avoiding random walk and the SAS models. Given the wealth of information that has been discovered for self-avoiding walks, and given the richness of the geometry of surfaces compared to the geometry of lines, it may be anticipated that SASs will prove to be a geometrical model of considerable interest.

Here we study only the simplest geometrical properties of SASs which are amenable to investigation by exact enumeration techniques. We have enumerated all SASs of up to ten plaquettes on the simple cubic lattice by writing a program based on the Martin (1972) algorithm. Because the number of surfaces grows very rapidly with the number of plaquettes (cf table 1), the enumeration to order 10 required approximately 80 h of CPU time on an IBM 3081 computer. Owing to the relatively small surface size accessible by enumeration, no attempt was made to distinguish between surfaces of different topological classes, e.g., closed versus open surfaces. The program used is a relatively straightforward generalisation of that used to enumerate lattice animals (see, e.g., Redner 1982 for a program listing), except that the elementary building blocks are plaquettes and the surface constraint that no more than two plaquettes can meet along a common edge is imposed. In addition to counting the number of n-plaquette surfaces, c_n, the statistics of the perimeter of free edges for each surface was counted so that the average perimeter length, $\langle p_n \rangle$, could be calculated. The mean-square radius of gyration of n-plaquette surfaces, $\langle R_g^2(n) \rangle$, was computed as well. For this purpose, we used the centre of each plaquette to define its coordinate position, and we assumed a lattice spacing of unity.

Table 1. Number of n-plaquette surfaces, the average perimeter and the average radius of gyration.

n	c_n	$\langle p_n \rangle$	$\langle R_g^2(n) \rangle$
1	3	4.0	0
2	18	6.0	0.666 667
3	146	7.890 411	1.041 096
4	1332	9.792 793	1.434 685
5	13 089	11.706 624	1.836 810
6	135 307	12.622 532	2.243 318
7	1 451 118	15.539 224	2.652 346
8	15 999 321	17.456 483	3.063 043
9	180 244 790	19.374 108	2.474 903
10	2065 946 265	21.291 919	2.887 598

In table 1, the number of n-plaquette surfaces are given. Straightforward analysis of the resulting series indicates that the asymptotic behaviour of the c_n is consistent with $c_n \sim \mu^n n^{\gamma-1}$, with $\mu = 13.2 \pm 0.2$, and $\gamma = 0.22 \pm 0.06$. Since the series are relatively short and not very well converged, it did not seem worthwhile to employ more sophisticated analysis methods. The value for γ may be compared with the exact result $\gamma = -0.5$ for lattice animals in three dimensions (Parisi and Sourlas 1981). This suggests that the surface constraint is relevant in three dimensions so that lattice animals and SASs belong to different universality classes.

It is worth noting that an exponential increase in c_n as a function of n has not yet been established rigorously. If the topology of the surface is fixed, i.e., the number of handles or holes in the surface is fixed, then it is known that the number of surfaces, subject to the aforementioned topological constraint, grows exponentially (Durhuus *et al* 1983). In fact, closed self-avoiding surfaces in three dimension are identical to the external perimeter of site lattice animals. Numerical simulations for this object (Sterling and Greensite 1983) indicate that c_n varies $\mu^n n^{\gamma-1}$, with $\mu \cong 1.7$ and $\gamma \cong 0.5$, rather different than our estimates for self-avoiding surfaces of all topologies. However, rigorous bounds on the number of SASs of all topologies is not known. Interestingly the number of n-plaquette random surfaces of all topologies grows extremely rapidly, as n^n (Weingarten 1980, Eguchi and Kawai 1982a, b). Questions related to the number of surfaces of a given topological type will require considerably more numerical work in order to arrive at definitive results.

In table 2, the number of n-plaquette surfaces are further classified according to the number of perimeter edges (both internal and external). From this information, the average perimeter of n-plaquette surfaces, $\langle p_n \rangle$, may be calculated, and this quantity is listed in the second column of table 1. Already by order 10, the data strongly suggest that $\langle p_n \rangle$ is proportional to n, with a proportionality constant approximately equal to 2.3. Such a proportionality constant is consistent with a surface structure which is very tree-like and stringy rather than a compact membrane-like structure.

Table 2. Perimeter distribution of self-avoiding surfaces. Listed are the number of n-plaquette surfaces which have a perimeter consisting of m lattice edges, $c_{n,m}$, in the form $c_{n,m}x^m$.

n	$c_{n,m}$
1	$3x^4$
2	$18x^6$
3	$8x^6+138x^8$
4	$12x^6+114x^8+1206x^{10}$
5	$6x^4+180x^8+1536x^{10}+11\,367x^{12}$
6	$1+48x^6+84x^8+2682x^{10}+19\,722x^{12}+112\,770x^{14}$
7	$24x^6+540x^8+2304x^{10}+38\,220x^{12}+248\,688x^{14}+1161\,342x^{16}$
8	$60x^6+483x^8+7032x^{10}+45\,498x^{12}+53\,3070x^{14}+3114\,414x^{16}+12\,298\,764x^{18}$
9	$30x^4+960x^8+8808x^{10}+105\,213x^{12}+779\,592x^{14}+7348\,344x^{16}+38\,900\,516x^{18}+$ $133\,101\,327x^{20}$
10	$3+240x^6+576x^8+16\,572x^{10}+156\,312x^{12}+1638\,774x^{14}+12\,502\,134x^{16}+100\,479\,600x^{18}$ $+485\,521\,476x^{20}+1465\,630\,578x^{22}$

Finally, the average radius of gyration of n-plaquette surfaces is given in the third column of table 1. An analysis of this quantity indicates that it diverges as $n^{2\nu}$, with $2\nu = 1.075 \pm 0.05$. For purposes of a qualitative comparison, the value of 2ν for lattice animals in three dimensions is exactly equal to unity (Parisi and Sourlas 1981).

In conclusion, self-avoiding random surfaces pose a new and interesting geometrical problem. By using enumeration methods, several basic configurational properties of SASs on the simple cubic lattice were calculated and their asymptotic behaviour was estimated. The numerical evidence suggests that SASs are in a new universality class, distinct from that of lattice animals. It is hoped that the present study, though rather limited in scope, will help stimulate more comprehensive and detailed studies of surfaces.

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